Row Reduction and Determinant

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The notes talk about two important manipulations of matrices – row reduction and determinant (Boas 3.2-3.3). Row reduction is closely related to coupled linear equations and the rank of a matrix. In general, a matrix does not correspond to a particular number. However, for a square matrix, there exists a useful number called *determinant*.

• Row reduction

Consider the set of coupled linear equations

$$\begin{aligned} 2x \qquad - z &= 2, \\ 6x + 5y + 3z &= 7, \\ 2x - y \qquad &= 4. \end{aligned}$$

From the above equations, we can define the *coefficient matrix* M and the *augmented matrix* A,

$$M = \begin{pmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix}. \tag{1}$$

By the standard technique of row reduction (details can be found in Boas), the augmented matrix can be bought into the canonical form,

$$A \to \begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{2}$$

where the coefficient matrix M is simplified into the identity matrix and the solution of the coupled linear equations is thus $(x, y, z) = (3/2, -1, 1)$.

• Rank of a matrix

The row reduction does not always to lead a unique solution for the coupled equations. For instance, consider the augmented matrix

$$A = \begin{pmatrix} 1 & -1 & 4 & 5 \\ 2 & -3 & 8 & 4 \\ 1 & -2 & 4 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 4 & 11 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & -20 \end{pmatrix}.$$

The last row means $0 \cdot z = -20$ which cannot be true for any finite value of z. Thus, no solution exists and the original equations are *inconsistent*. For convenience, we introduce the notion of *rank*. The rank of a matrix is the number of nonzero rows remaining after row reduction. For the example shown here, rank of A is 3 but the rank of M is 2. In general, the possible results are listed below,

- If rank $M <$ rank A, the equations are inconsistent and there is no solution at all.
- If rank $M =$ rank $A = n$ (the number of unknowns), there exists one and only one solution.
- If rank $M =$ rank $A = R < n$, R unknowns can be solved in terms of the remaining $n - R$ unknowns.

Note that rank M cannot exceed rank A from their definitions.

• Determinant

Before explaining how to compute the determinant for a square matrix, it is helpful to introduce the Levi-Civita symbol for permutations,

$$\epsilon_{i_1 i_2 \ldots i_n} = \begin{cases} 1, & \text{if } (i_1 i_2 \ldots i_n) \text{ is an even permutation of } (12\ldots n), \\ -1, & \text{if } (i_1 i_2 \ldots i_n) \text{ is an odd permutation of } (12\ldots n), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

It is clear that the Levi-Civita symbol is nonzero when all indices $(i_1 i_2 \ldots i_n)$ are distinct and thus a permutation of the sequence $(12\ldots n)$. Take $n = 3$ as an example,

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} &= 1, \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} &= -1, \\ \epsilon_{111} = \epsilon_{112} = \epsilon_{113} = \ldots &= 0. \end{aligned}$$

The determinant of a square matrix can be expressed in terms of the Levi-Civita symbol,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} = \sum_{i_1 i_2 \ldots i_n} \epsilon_{i_1 i_2 \ldots i_n} a_{1 i_1} a_{2 i_2} \ldots a_{n i_n}. \tag{4}$$

The above expression may look terrifying but can be useful from time to time. To warm up, we can work from the simplest $n = 2$ determinant,

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \sum_{i_1 i_2} \epsilon_{i_1 i_2} a_{1 i_1} a_{2 i_2} \\ &= \epsilon_{11} a_{11} a_{21} + \epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{12} a_{21} + \epsilon_{22} a_{12} a_{22} \\ &= a_{11} a_{22} - a_{12} a_{21}. \end{aligned}$$

The result is the same as using the trick of "drawing diagonal lines" in high school math. However, be aware that the trick fails when $n \geq 4$.

A useful trick to evaluate the determinant is by *Laplace decomposition*,

$$\det A = \sum_j a_{ij} C_{ij} = \sum_j (-1)^{i+j} a_{ij} M_{ij}, \tag{5}$$

where C_{ij} is the cofactor and M_{ij} is the minor of a_{ij} with the sign relation $C_{ij} = (-1)^{i+j} M_{ij}$. It is important to emphasize that there is no summation on the index i. The best way to learn the minors and the cofactors of a matrix is through concrete examples. For instance, in the determinant

$$\begin{vmatrix} 1 & -5 & 2 \\ 7 & 3 & 4 \\ 2 & 1 & 5 \end{vmatrix}$$

the minor and the cofactor of the element $a_{23} = 4$ are

$$M_{23} = \begin{vmatrix} 1 & -5 \\ 2 & 1 \end{vmatrix}, \qquad C_{23} = (-1)^{2+3} M_{23} = -M_{23}.$$

Making use of the properties for the Levi-Civita symbol, we can show why Laplace decomposition works for the determinant. Suppose we choose the the first row to expand the determinant,

$$\begin{aligned} \det A &= \sum_{i_1 i_2 \ldots i_n} \epsilon_{i_1 i_2 \ldots i_n} a_{1 i_1} a_{2 i_2} \ldots a_{n i_n} = a_{11} \sum_{i_2 \ldots i_n} \epsilon_{1 i_2 \ldots i_n} a_{2 i_2} a_{3 i_3} \ldots a_{n i_n} \\ &\quad + a_{12} \sum_{i_2 \ldots i_n} \epsilon_{2 i_2 \ldots i_n} a_{2 i_2} a_{3 i_3} \ldots a_{n i_n} + \ldots + a_{1n} \sum_{i_2 \ldots i_n} \epsilon_{n i_2 \ldots i_n} a_{2 i_2} a_{3 i_3} \ldots a_{n i_n}. \end{aligned}$$

Since the Levi-Civita symbol is nonzero only when all induces are distinct,

$$\sum_{i_2 \ldots i_n = 1}^{n} \epsilon_{1 i_2 \ldots i_n} a_{2 i_2} a_{3 i_3} \ldots a_{n i_n} = M_{11},$$

Similarly, the second summation is

$$\sum_{i_2 \ldots i_n = 1}^{n} \epsilon_{2 i_2 \ldots i_n} a_{2 i_2} a_{3 i_3} \ldots a_{n i_n} = -M_{12}.$$

The minus sign arises from the relation $\epsilon_{2 i_2 i_3 \ldots i_n} = -\epsilon_{i_2 2 i_3 \ldots i_n}$. It is also straightforward to check the sign for the third summation, $\epsilon_{3 i_2 i_3 \ldots i_n} = +\epsilon_{i_2 i_3 3 \ldots i_n}$. Therefore, the determinant can be written as

$$\det A = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} + \ldots + (-1)^{1+n} a_{1n} M_{1n}. \tag{6}$$

This is just the Laplace decomposition we mentioned before.