Linear Operators
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The notes cover linear operators and discuss linear independence of functions (Boas 3.7-3.8).

• Linear operators

An operator maps one thing into another. For instance, the ordinary functions are operators mapping numbers to numbers. A linear operator satisfies the properties,

\[ O(A + B) = O(A) + O(B), \quad O(kA) = kO(A), \]  

where \( k \) is a number. As we learned before, a matrix maps one vector into another. One also notices that

\[ M(r_1 + r_2) = Mr_1 + Mr_2, \quad M(kr) = kMr. \]

Thus, matrices are linear operators.

• Orthogonal matrix

The length of a vector remains invariant under rotations,

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} \left( \begin{pmatrix} x' \\ y' \end{pmatrix} \right) = \begin{pmatrix} x \\ y \end{pmatrix} M^T M \begin{pmatrix} x \\ y \end{pmatrix}. \]

The constraint can be elegantly written down as a matrix equation,

\[ M^T M = M M^T = 1. \]  

(2)

In other words, \( M^T = M^{-1} \). For matrices satisfy the above constraint, they are called orthogonal matrices. Note that, for orthogonal matrices, computing inverse is as simple as taking transpose – an extremely helpful property for calculations.

From the product theorem for the determinant, we immediately come to the conclusion \( \det M = \pm 1 \). In two dimensions, any \( 2 \times 2 \) orthogonal matrix with determinant 1 corresponds to a rotation, while any \( 2 \times 2 \) orthogonal
matrix with determinant $-1$ corresponds to a reflection about a line. Let’s come back to our good old friend – the rotation matrix,

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad R^T = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (3)$$

It is straightforward to check that $R^T R = R R^T = 1$.

You may wonder why we call the matrix “orthogonal”? What does it mean that a matrix is orthogonal? (to what?!) Here comes the charming reason for the name. Writing down the product $R^T R$ explicitly,

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

we realize that an orthogonal matrix contains a complete bases of orthogonal vectors in the same dimensions!

### Rotations and reflections in 2D

Consider the rotation matrix and the reflection about the $x$-axis (also called parity operator in the $y$-direction),

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

We can construct two operators by combining $R(\theta)$ and $P_y$ in different orders,

$$C = R(\theta) P_y, \quad D = P_y R(\theta). \quad (6)$$

One can check that $\det C = \det D = -1$ and they do not correspond to the usual rotations. Carrying out the matrix multiplication, the operator $C$ in explicit matrix form is

$$C = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \quad (7)$$

To figure what the operator do, we can act $C$ on unit vectors along $x$- and $y$-directions,

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}.$$
Plotting out the mappings, one can see that $C$ corresponds to a reflection about the line at $\theta/2$. While the geometric picture is nice, it is also comforting to know about the algebraic approach,

$$Cr = r \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (8)$$

After some algebra, the above matrix equation gives the relation for the reflection line,

$$\frac{y}{x} = \frac{\sin(\theta/2)}{\cos(\theta/2)}.$$

This is exactly what we expected. Now we turn to the other operator $D$,

$$D = PyR(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}. \quad (9)$$

You may have guessed that $D$ corresponds to a reflection about some line – this is indeed true. Absorbing the minus sign into the sin function, we come to the identity

$$PyR(\theta) = R(-\theta)Py = R^{-1}(\theta)Py. \quad (9)$$

Thus, $D$ corresponds to a reflection about the line at $-\theta/2$.

**Rotations and reflections in 3D**

We can generalize the discussions to three dimensions. Any $3 \times 3$ orthogonal matrices with determinant 1 can be brought into the standard form by choosing the rational axis to coincide with the $z$-axis,

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (10)$$

Similarly, Any $3 \times 3$ orthogonal matrices with determinant $-1$ can be brought into the standard form,

$$\tilde{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (11)$$
and corresponds to a rotation about the (appropriate) $z$-axis followed by a reflection through the $xy$-plane. An example will help to digest the notation,

$$ L = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

First of all, $\det L = -1$ and thus corresponds to an improper rotation (rotation + reflection). We can find out the normal vector for the reflection plane,

$$ Ln = -n \rightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}. $$

Or, we can take a different view and try to figure out the equation for the plane directly,

$$ Lr = r \rightarrow \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. $$

Both methods give the reflection plane $x + y = 0$ and explains the action of the operator $L$.

**Wronskian for linear independence**

Following similar definition for vectors, we say that a set of functions is linearly dependent if some linear combinations of them give identical zero,

$$ k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x) = 0, \quad (12) $$

where $k_1^2 + k_2^2 + ... + k_n^2 \neq 0$. Taking derivatives of the above equation, we can cook up a complete set of equations,

$$ k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x) = 0, $$

$$ k_1 f_1'(x) + k_2 f_2'(x) + ... + k_n f_n'(x) = 0, $$

$$ \vdots $$

$$ k_1 f_1^{(n-1)}(x) + k_2 f_2^{(n-1)}(x) + ... + k_n f_n^{(n-1)}(x) = 0. $$
If we can find non-trivial solutions for \((k_1, k_2, ..., k_n)\), the functions are linearly dependent. From previous lectures, we know that it amounts to require

\[
W(f_1, f_2, ..., f_n) \equiv \left| \begin{array}{cccc}
  f_1 & f_2 & ... & f_n \\
  f'_1 & f'_2 & ... & f'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(n-1)}_1 & f^{(n-1)}_2 & ... & f^{(n-1)}_n \\
\end{array} \right| = 0, \tag{13}
\]

where \(W(f_1, f_2, ..., f_n)\) is the Wronskian. It is important to emphasize that “dependent functions” implies \(W = 0\), but \(W = 0\) does not necessarily imply the functions are linearly dependent.