Linear Operators

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The notes cover linear operators and discuss linear independence of functions (Boas 3.7-3.8).

• Linear operators

An operator maps one thing into another. For instance, the ordinary functions are operators mapping numbers to numbers. A linear operator satisfies the properties,

$$O(A+B) = O(A) + O(B), \qquad O(kA) = kO(A), \tag{1}$$

where k is a number. As we learned before, a matrix maps one vector into another. One also notices that

$$M(\boldsymbol{r}_1 + \boldsymbol{r}_2) = M\boldsymbol{r}_1 + M\boldsymbol{r}_2, \qquad M(k\boldsymbol{r}) = kM\boldsymbol{r}.$$

Thus, matrices are linear operators.

• Orthogonal matrix

The length of a vector remains invariant under rotations,

$$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} M^T M \begin{pmatrix} x \\ y \end{pmatrix}.$$

The constraint can be elegantly written down as a matrix equation,

$$M^T M = M M^T = \mathbf{1}. \tag{2}$$

In other words, $M^T = M^{-1}$. For matrices satisfy the above constraint, they are called *orthogonal* matrices. Note that, for orthogonal matrices, computing inverse is as simple as taking transpose – an extremely helpful property for calculations.

From the product theorem for the determinant, we immediately come to the conclusion $\det M = \pm 1$. In two dimensions, any 2×2 orthogonal matrix with determinant 1 corresponds to a rotation, while any 2×2 orthogonal matrix with determinant −1 corresponds to a reflection about a line. Let's come back to our good old friend – the rotation matrix,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3}$$

It is straightforward to check that $R^T R = RR^T = \mathbf{1}$.

You may wonder why we call the matrix "orthogonal"? What does it mean that a matrix is orthogonal? (to what?!) Here comes the charming reason for the name. Writing down the product $R^T R$ explicitly,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}_1\cdot\boldsymbol{v}_1 & \boldsymbol{v}_1\cdot\boldsymbol{v}_2 \\ \boldsymbol{v}_2\cdot\boldsymbol{v}_1 & \boldsymbol{v_2}\cdot\boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

we realize that an orthogonal matrix contains a complete bases of orthogonal vectors in the same dimensions!

• Rotations and reflections in 2D

Consider the rotation matrix and the reflection about the x-axis (also called parity operator in the y-direction),

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad P_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

We can construct two operators by combining $R(\theta)$ and P_y in different orders,

$$C = R(\theta)P_y, \qquad D = P_y R(\theta). \tag{6}$$

One can check that $\det C = \det D = -1$ and they do not correspond to the usual rotations. Carrying out the matrix multiplication, the operator C in explicit matrix form is

$$C = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{7}$$

To figure what the operator do, we can act C on unit vectors along x- and y-directions,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

Plotting out the mappings, one can see that C corresponds to a reflection about the line at $\theta/2$. While the geometric picture is nice, it is also comforting to know about the algebraic approach,

$$C\boldsymbol{r} = \boldsymbol{r} \quad \rightarrow \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{8}$$

After some algebra, the above matrix equation gives the relation for the reflection line,

$$\frac{y}{x} = \frac{\sin(\theta/2)}{\cos(\theta/2)}.$$

This is exactly what we expected. Now we turn to the other operator D,

$$D = P_y R(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}.$$

You may have guessed that D corresponds to a reflection about some line – this is indeed true. Absorbing the minus sign into the sin function, we come to the identity

$$P_y R(\theta) = R(-\theta) P_y = R^{-1}(\theta) P_y. \tag{9}$$

Thus, D corresponds to a reflection about the line at $-\theta/2$.

• Rotations and reflections in 3D

We can generalize the discussions to three dimensions. Any 3×3 orthogonal matrices with determinant 1 can be brought into the standard form by choosing the rational axis to coincide with the z-axis,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

Similarly, Any 3×3 orthogonal matrices with determinant -1 can be brought into the standard form,

$$\tilde{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11}$$

and corresponds to a rotation about the (appropriate) z-axis followed by a reflection through the xy-plane. An example will help to digest the notaion,

$$L = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

First of all, $\det L = -1$ and thus corresponds to an improper rotation (rotation + reflection). We can find out the normal vector for the reflection plane,

$$L\boldsymbol{n} = -\boldsymbol{n} \quad \rightarrow \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = - \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}.$$

Or, we can take a different view and try to figure out the equation for the plane directly,

$$L\boldsymbol{r} = \boldsymbol{r} \quad \rightarrow \quad \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Both methods give the reflection plane $x + y = 0$ and explains the action of the operator L.

• Wronskian for linear independence

Following similar definition for vectors, we say that a set of functions is linearly dependent if some linear combinations of them give identical zero,

$$k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x) = 0, \tag{12}$$

where $k_1^2 + k_2^2 + ... + k_n^2 \neq 0$. Taking derivatives of the above equation, we can cook up a complete set of equations,

$$\begin{aligned} k_1 f_1(x) + k_2 f_2(x) + \quad & ... \quad + k_n f_n(x) = 0, \\ k_1 f_1'(x) + k_2 f_2'(x) + \quad & ... \quad + k_n f_n'(x) = 0, \\ & \vdots \\ k_1 f_1^{(n-1)}(x) + k_2 f_2^{(n-1)}(x) + \quad & ... \quad + k_n f_n^{(n-1)}(x) = 0. \end{aligned}$$

If we can find non-trivial solutions for $(k_1, k_2, ..., k_n)$, the functions are linearly dependent. From previous lectures, we know that it amounts to require

$$W(f_1, f_2, ..., f_n) \equiv \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = 0, \qquad (13)$$

where $W(f_1, f_2, ..., f_n)$ is the Wronskian. It is important to emphasize that "dependent functions" implies $W = 0$, but $W = 0$ does not necessarily imply the functions are linearly dependent.