CS 2336
Discrete Mathematics

Lecture 12

Sets, Functions, and Relations: Part IV

Outline

- Equivalence Relations
- Partial Orderings

Equivalence Relations

Equivalence Relations

- A relation may have more than one properties

A binary relation R on a set A is an equivalence relation if it is reflexive, symmetric, and transitive

- Suppose that A = cities in a country X

$$R = \{ (x, y) \mid x \text{ can reach } y \text{ by roads } \}$$

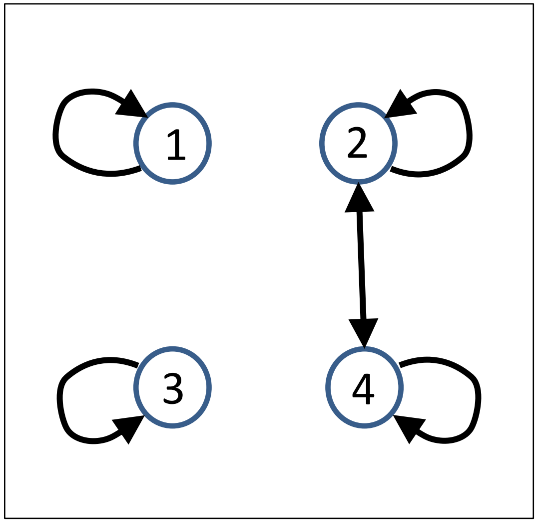
Is R always an equivalence relation, if

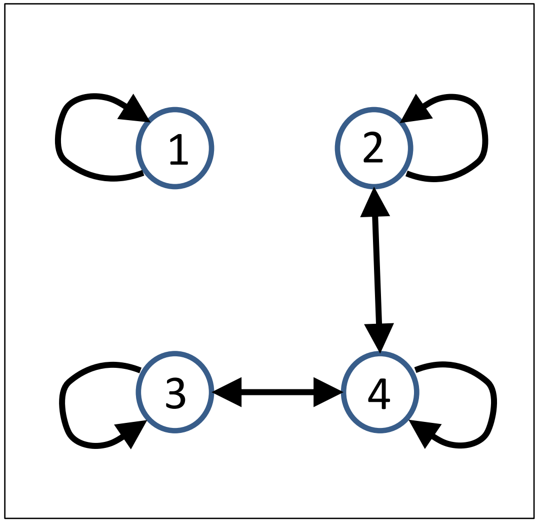
(1) roads are always two-ways ?

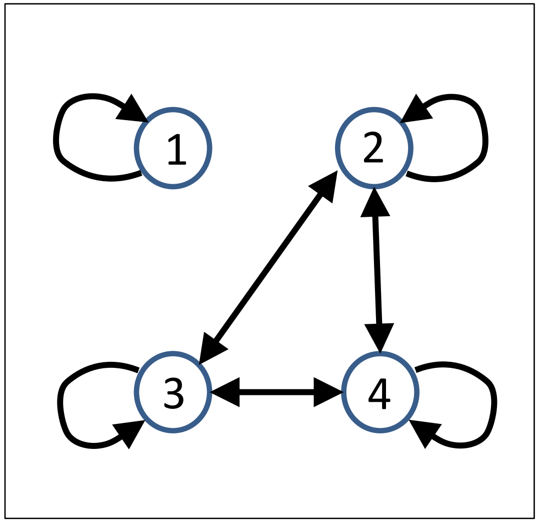
(2) some roads may be one-way ?

Equivalence Relations

- Which of the following are equivalence relations ?







Equivalence Relations

- Suppose a binary relation R contains the pairs (a, b) as defined in the following cases

  In which of the following cases will R be an equivalence relation ?

  1. (a, b) such that a – b is an integer ; $a, b \in \Re$
  2. (a, b) such that $|a - b| < 1$; $a, b \in \Re$
  3. (a, b) such that a – b is divisible by 3 ; $a, b \in Z$
  4. (a, b) such that a divides b ; $a, b \in Z^+$

Equivalence Classes

- Let R be an equivalence relation on A, and
  x be an item in A
- If $(x, w) \in R$, we say w is related to x, and
  we denote this by xRw

The equivalence class of x (with respect to R) is the set of items related to x :

$$[x]_R = \{ w \mid (x, w) \in R \}.$$

Equivalence Relations

- Example 1 :

  R = { (a, b) | a – b is divisible by 3, a, b $\in \mathbb{Z}$ }

➔ $[-1]_R$ = { …, -7, -4, -1, 2, 5, 8, 11, … }

$[0]_R$ = { …, -9, -6, -3, 0, 3, 6, 9, … }

$[1]_R$ = { …, -8, -5, -2, 1, 4, 7, 10, … }

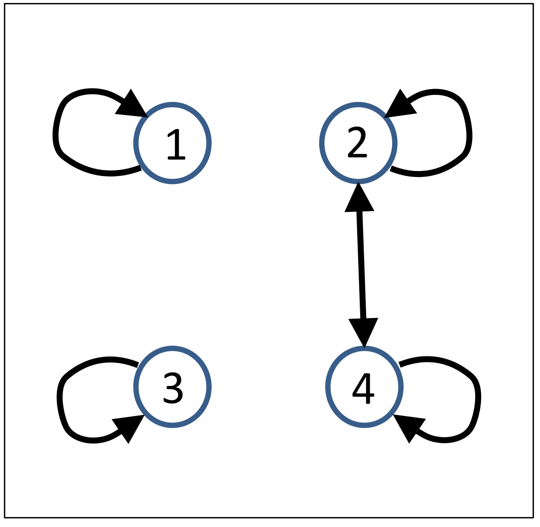
$[2]_R$ = { …, -7, -4, -1, 2, 5, 8, 11, … }

$[7]_R$ = { …, -8, -5, -2, 1, 4, 7, 10, … }

Equivalence Classes

- Example 2 :

  R = equivalence relation corresponding to the directed graph below



$[1]_R = \{ 1 \}$

$[2]_R = \{ 2, 4 \}$

$[3]_R = \{ 3 \}$

$[4]_R = \{ 2, 4 \}$

Equivalence Classes

- Let R be an equivalence relation on A, and

  x and y be items in A

Theorem: The equivalence classes of x and y are either the same, or disjoint. That is, either

$$[x]_R = [y]_R \quad \text{or} \quad [x]_R \cap [y]_R = \varnothing$$

- Proof (see next page)

Equivalence Classes

Proof :

Case 1: x is related to y

➔ xRy ➔ yRx (why?)

➔ When xRz, we have yRz (since yRx and xRz)

➔ $[x]_R = [y]_R$

Case 2: x is not related to y

➔ Suppose on the contrary that $[x]_R \cap [y]_R \neq \varnothing$

➔ There exists some z such that xRz and yRz

➔ xRz and zRy (why?) ➔ xRy ➔ contradiction !

Partition of a Set

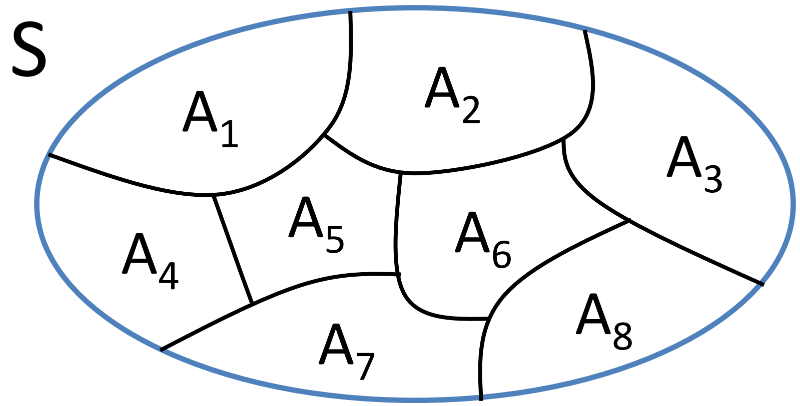
- Let S be a set

A partition of S is a collection of disjoint subsets of S such that their union is S. In other words, the collection of subsets $A_1, A_2, \ldots, A_k$ forms a partition of S, if and only if

(i) $A_1 \cup A_2 \cup \ldots \cup A_k = S$, and

(ii) $A_i \cap A_j = \emptyset$ for all $i \neq j$

Partition of a Set



- Let R be an equivalence relation on a set A

Theorem :

The equivalence classes of R form a partition of A

Partition of a Set

- In fact, the converse of the previous theorem is also true

Theorem :

Let A_1, A_2, …, A_k be disjoint subsets that form a partition of a set S. There exists an equivalence relation R on S such that has the sets A_1, A_2, …, A_k as its equivalence classes

- It is easy to prove these two theorems. How ?

Partial Orderings

Partial Orderings

- Another important relation is the following:

> A binary relation $\propto$ on a set A is a partial ordering if it is reflexive, antisymmetric, and transitive

- A set S with a partial ordering $\propto$ is called a partial ordered set (or poset)

Partial Orderings

- Suppose a binary relation $\propto$ contains the pairs (a, b) as defined in the following cases

  In which cases will $\propto$ be a partial ordering ?

  1. (a, b) such that $a \leq b$; $a, b \in \Re$
  2. (a, b) such that $a < b$; $a, b \in \Re$
  3. (a, b) such that $a - b$ is divisible by 3 ; $a, b \in Z$
  4. (a, b) such that a divides b ; $a, b \in Z^+$

Partial Orderings

- Let S be a set of people

  Suppose a binary relation $\propto$ contains the pairs as defined in the following cases

  In which cases will $\propto$ be a partial ordering ?

  1. (A, B) such that $A \subseteq B$; $A, B \in 2^S$
  2. (x, y) such that x is older than y ; $x, y \in S$
  3. (x, y) such that x is not older than y ; $x, y \in S$

Partial Orderings

- Two items x and y in a poset $(S, \propto)$ are comparable if either $x \propto y$ or $y \propto x$
- Otherwise, x and y are incomparable

- Example :

  Consider the poset $(\mathrm{Z}^+, |)$, where a | b means a divides b

  1. Are the integers 3 and 9 comparable ?
  2. Are 5 and 7 comparable ?

Hasse Diagrams

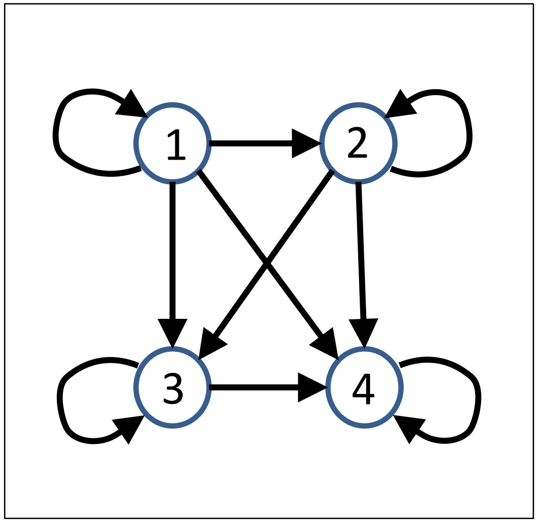
- Since partial orderings is a binary relation, it can be represented by a directed graph
- However, many edges can be omitted, because such an ordering must be reflexive and transitive
- Also, we may order the vertices in the graph in a 'vertical' manner, such that all edges are pointing from low to high

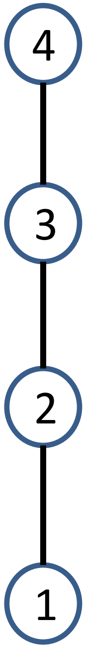
  ➔ directions on an edge can be omitted

- See the next two pages for examples

Hasse Diagrams

- Consider the poset ({ 1, 2, 3, 4 }, $\leq$)

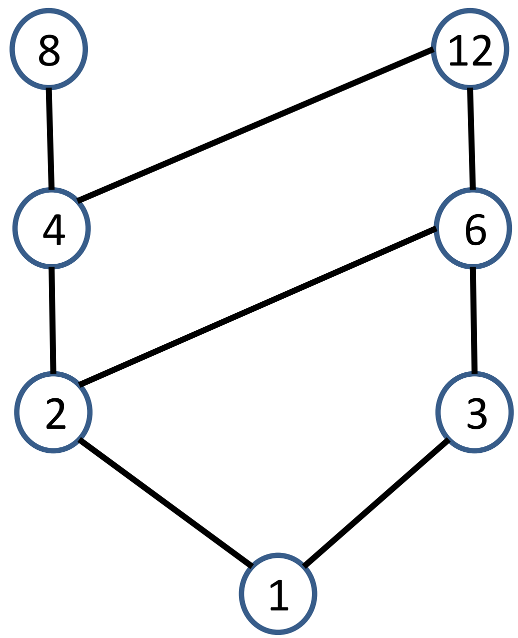




original representation Hasse diagram

Hasse Diagrams

- Consider the poset $(\{ 1, 2, 3, 4, 6, 8, 12 \}, \; | \;)$



Hasse diagram

Hasse Diagrams

- To summarize, the following are the steps to obtain a Hasse diagram :
  1. Remove all the self loops
  2. Remove all the edges that must be present due to transitivity
  3. Arrange all edges to point upwards
  4. Do not show directions on the edges

Maximal and Minimal Items

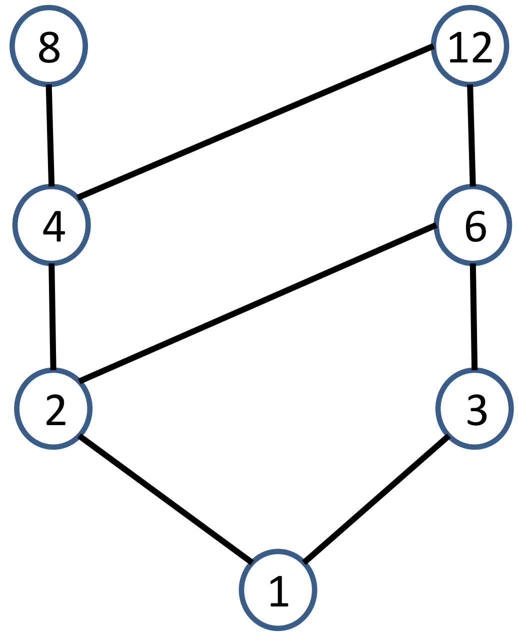
- One important concept of a poset is the following:

> An item x in a poset $(S, \propto)$ is maximal if there is no item y with $x \neq y$ and $x \propto y$
>
> That is, in the Hasse diagram, x is not connected to anything above x

- Similarly, we define minimal item to be an item y where there is no item x with $x \neq y$ and $x \propto y$

Maximal and Minimal Items

- Which items are maximal ? Which are minimal ?



Hasse diagram

Maximal and Minimal Items

- We have the following observation :

> Every finite nonempty poset (S, $\propto$) has as at least one minimal item

- Proof : We give a method to find a minimal item. Pick any item x. Either x is minimal, then we are done. Else, we get some y ($x \neq y$) such that $y \propto x$, and repeat the process. Each time, we must either get a minimal item, or test a new item in S. Since S is finite, this process must end.

Total Orderings

If $(S, \prec)$ is a poset such that every two items in S are comparable, we call S a totally ordered set

- Which of the following is a totally ordered set ?
  1. $(\Re, \geq)$
  2. $(2^Q, \subseteq)$, Q = a set of people
  3. $(Z^+, |)$

Compatible Orderings

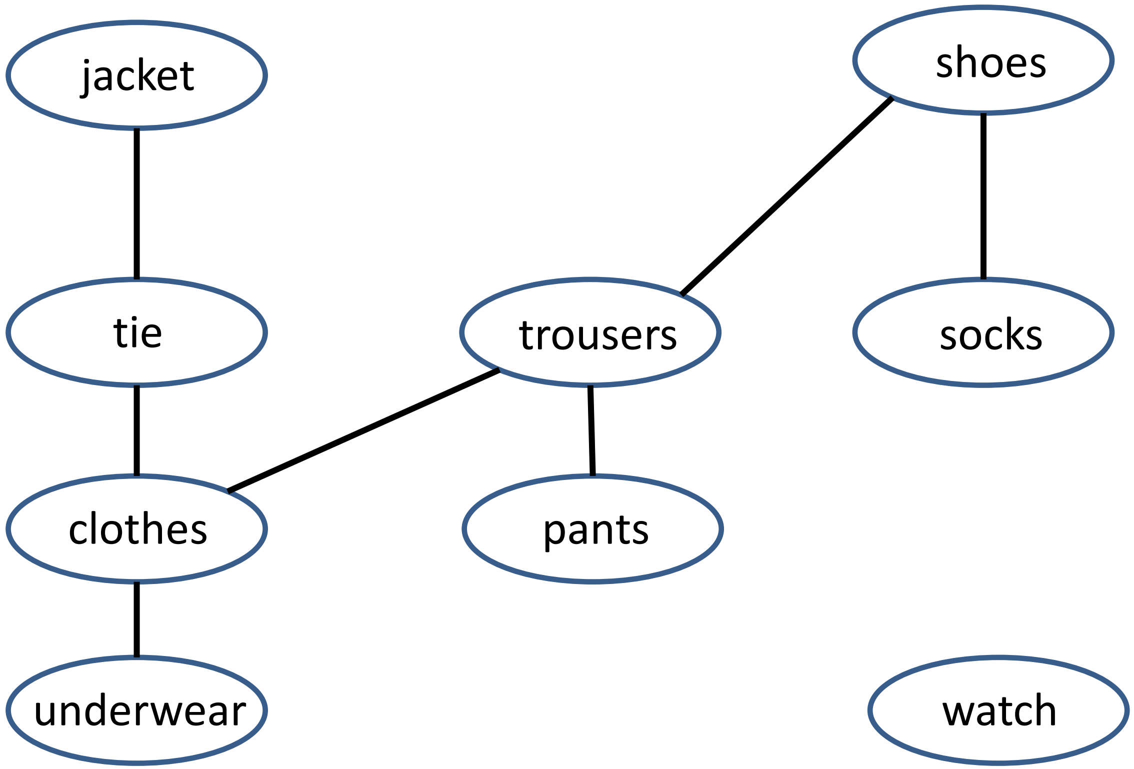
Let (S, $\propto$) be a poset. A total ordering (S, $\prec$) is compatible with (S, $\propto$), if for any items x and y,

$$x \propto y \quad \text{implies} \quad x \prec y$$

Equivalently, $\propto$ is a subset of $\prec$

- Every finite poset has at least one compatible total ordering ; Constructing one such ordering is called the topological sorting problem

Compatible Orderings



A compatible total ordering :

underwear ≺ clothes ≺ pants ≺ trousers ≺ socks ≺ tie ≺ shoes ≺ jacket ≺ watch

Compatible Orderings

- Given a finite poset (S, $\propto$), the following shows how to obtain a compatible total ordering :

1. Get a minimal item x
2. Denote x as the smallest item
3. Remove the minimal item x from S, and recursively find the compatible total ordering
4. Place the remaining items, from small to large as defined in 3, after the item x

Upper and Lower Bounds

- We define the following terms :

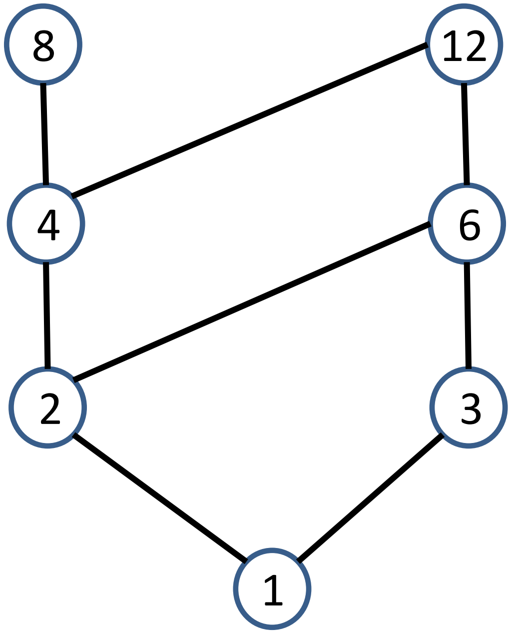
> Let $(S, \propto)$ be a poset. For items x and y, if there is an item z such that $x \propto z$ and $y \propto z$, then z is called an upper bound of x and y

- Similarly, for items x and y, if there is an item w such that $w \propto x$ and $w \propto y$, then w is called a lower bound of x and y

Upper and Lower Bounds

- Consider the poset as shown on the right

- Which items are :
  1. Upper bounds of 2 and 3 ?
  2. Lower bounds of 2 and 3 ?
  3. Upper bounds of 6 and 8 ?
  4. Lower bounds of 6 and 8 ?



LUB and GLB

- We further define the following terms :

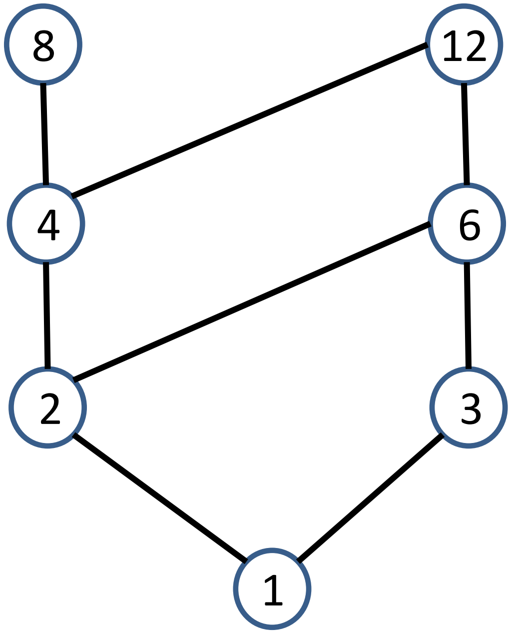
An item z is a least upper bound (LUB) for items x and y, if z is an upper bound of x and y, and no other upper bound d of x and y exists with $d \propto z$

- Similarly, an item w is called a greatest lower bound (GLB) of x and y, if w is a lower bound of x and y, and no other lower bound c of x and y exists with $w \propto c$

LUB and GLB

- Consider the poset as shown on the right

- Which items are :
  1. LUB of 2 and 3 ?
  2. GLB of 2 and 3 ?
  3. LUB of 6 and 8 ?
  4. GLB of 6 and 8 ?



Lattices

- Lattice is a type of poset with special properties :

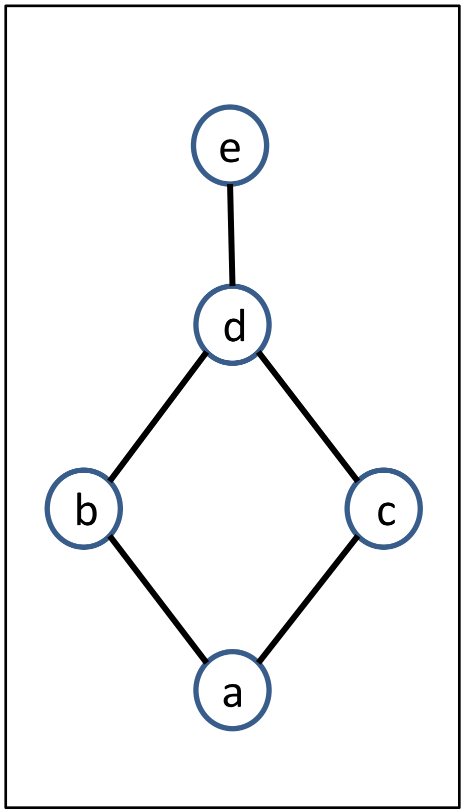
A poset $(S, \propto)$ is a lattice if for any items x and y, there is a unique LUB and a unique GLB

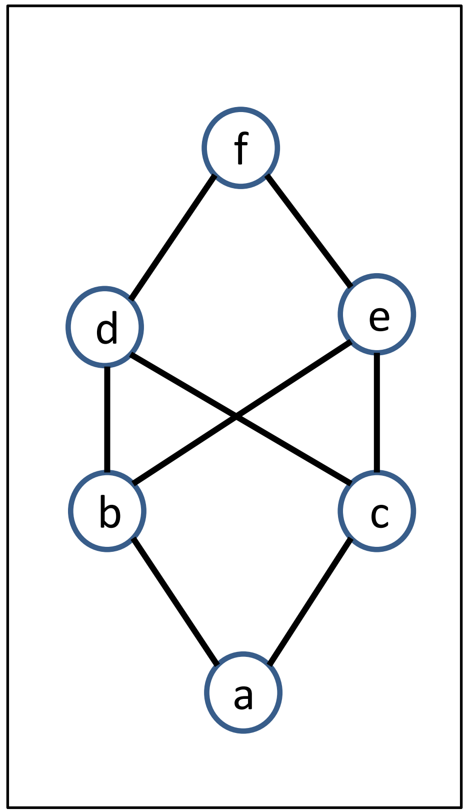
- Examples :

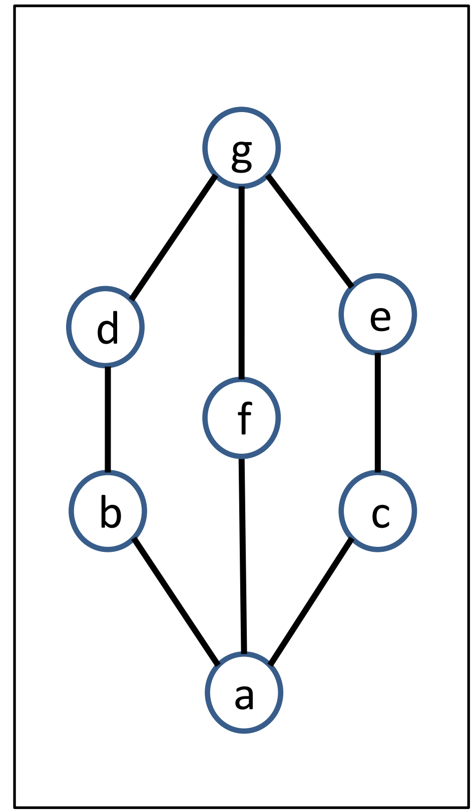
1. Is $(\mathrm{Z}^{+}, |)$ a lattice ?
2. Is $(2^S, \subseteq)$ a lattice ?

Lattices

- Which of the following are lattices ?







Lattices

- Lattice is related to information flow and Boolean algebra, and has many properties

- Examples :

1. LUB and GLB are commutative and associative :

$$LUB(x, y) = LUB(y, x)$$

$$LUB(x, LUB(y, z)) = LUB(LUB(x, y), z)$$

2. Duality (upside-down is also a lattice):

   If $(S, \propto)$ is a lattice, then $(S, \propto')$ is also a lattice, where $x \propto y$ if and only if $y \propto' x$

Principle of Duality

- Duality is an important concept that appears in many places, and can be used to create results
- Example :

  Cars in Taiwan or US travel on the right side ;

  On a highway, left lanes are passing lanes

  ➔ We can construct a system, switching left and right, so that cars travel on the left side, and on a highway, right lanes are passing lanes

  ➔ Such a system is indeed used in UK, HK, Japan

An Interesting Example

- The following is a typical conversation between the people in the Land of Gentlemen, inside the story Flowers in the Mirror

Customer :	I want to buy this vase. It is most beautiful.
Storekeeper :	The vase does not really look that good. Please note that there are some defects in the paint job.
Customer :	You would not call these defects, would you? The vase is a perfect piece of art work. How much do you want for it ?
Storekeeper :	Usually I will sell a vase like this for \$10. Since you have been a good customer for many years, I will sell it for \$15.
Customer :	\$15 is too low. I cannot allow myself to take advantage of you. How about \$20?
Storekeeper :	I would be making a good profit at \$15. Since you insist, let us agree on \$16.