Goodness-of-Fit Test

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1. Chi-Square Goodness-of-Fit Test

2. Kolmogorov-Smirnov (K-S) Test

Chi-Square Goodness-of-Fit Test

- Goal: To decide if iid radom variables ($\{X_i\}_{i=1}^n = \{X_1, X_2, \ldots, X_n\}$) follow a specific dis., say F_X.
- Notation: (1) Divided data into b bins.
  (2) m is the no. of the estimated popu. parameters.
  (3) e_i, No. of expected observations
  (4) o_i, No. of observed observations

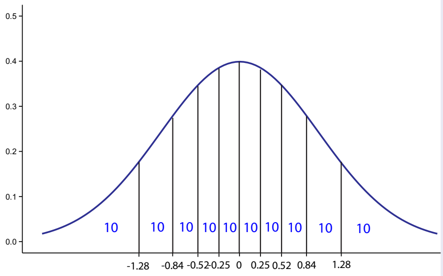
Chi-Square Goodness-of-Fit Test

$\mathrm{H}_0 : X \sim F_X$; $\mathrm{H}_1 : X \nsim F_X$

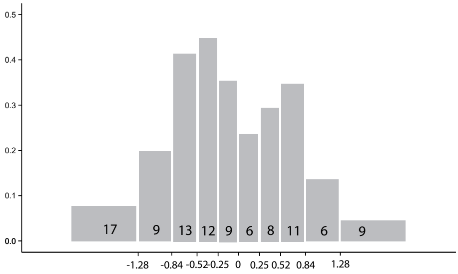
- Statistic (統計量): $C = \sum_{i=1}^b \frac{(e_i - O_i)^2}{e_i}$ (抽樣前); 統計值: c (抽樣後)
- The Statistic C follows a chi-squared dis. with d.f. $(b - m - 1)$
- p-value: P($C > c|X \sim F_X$), where $C \sim \chi^2$ with d.f. $(b - m - 1)$
- Let $\alpha = 0.05$. Reject H_0 if p-value < 0.05. i.e., $C > \chi^2_{0.05}(b - m - 1)$

Chi-Square Test Via PDF

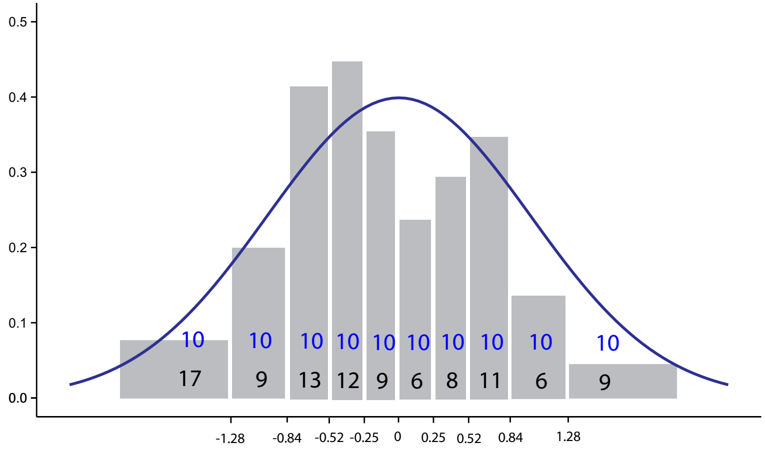
PDF, f_X



Estimated PDF, $\hat{f}_X$



Chi-Square Test Via PDF - Conti.



Kolmogorov-Smirnov (K-S) Test

- Goal: To decide if a continuous sample ($\{X_i\}_{i=1}^n = \{X_1, X_2, \ldots, X_n\}$) comes from a population with a specific dis., say F_X.
- Notation: Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered statistics from smallest to the largest.

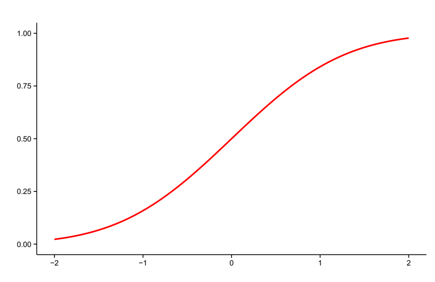
K-S Test (Chakravart, Laha, and Roy, 1967)

$\mathrm{H}_0 : X \sim F_X; \mathrm{H}_1 : X \nsim F_X$

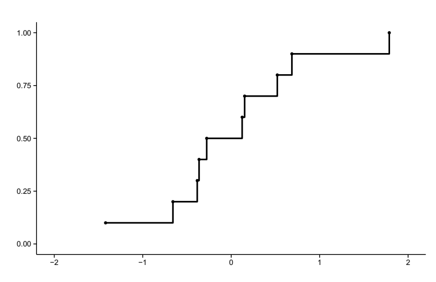
- Statistic (統計量): $D = \max_{1 \le i \le n} |F_X(X_{(i)}) - \hat{F}_X(X_{(i)})|$, where $\hat{F}_X(X_{(i)}) = i/n$
- Real value (統計值): $d = \max_{1 \le i \le n} |F_X(x_{(i)}) - \hat{F}_X(x_{(i)})|$, where $\hat{F}_X(x_{(i)}) = i/n$
- $\sqrt{n}D$ converges to "Kolmogorov distribution"
- $K \sim$ Kolmogorov distribution: $P(K \leqslant y) = 1 - 2\sum_{i=1}^{\infty}(-1)^{i-1}e^{-2i^2y^2}$
- p-value: P($D > d | X \sim F_X$), where $D = K/\sqrt{n} \equiv K^*$, ($K^*$ Check K-S Table)
- Let $\alpha = 0.05$. Reject H_0 if p-value < 0.05. i.e., $D > K_{\alpha}^*(n)$

K-S Test via CDF

CDF, F_X

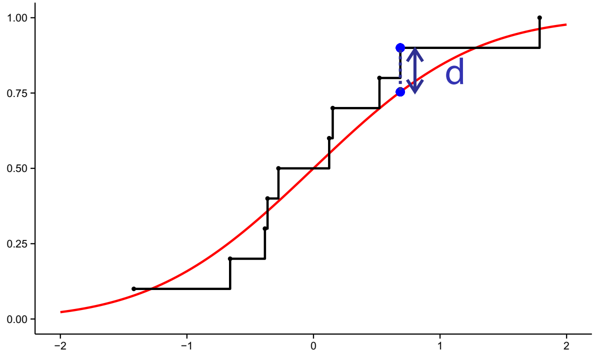


Estimated CDF, $\hat{F}_X$



K-S Test via CDF - Conti.

- $d = \max_{1 \le i \le n} |F_X(x_{(i)}) - \hat{F}_X(x_{(i)})|$, where $\hat{F}_X(x_{(i)}) = i/n$



K-S Table, $K^*_\alpha(n)$

	α, P(Type I error)				
n	0.01	0.05	0.1	0.15	0.2
5	0.669	0.565	0.510	0.474	0.446
10	0.490	0.410	0.368	0.342	0.322
15	0.404	0.338	0.304	0.283	0.266
20	0.356	0.294	0.264	0.246	0.231
25	0.320	0.270	0.220	0.200	0.190
30	0.290	0.240	0.220	0.200	0.190
35	0.270	0.230	0.210	0.190	0.180
40	0.250	0.210	0.190	0.180	0.170
45	0.240	0.200	0.180	0.170	0.160
50	0.230	0.190	0.170	0.160	0.150
over 50	$\frac{1.63}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.14}{\sqrt{n}}$	$\frac{1.07}{\sqrt{n}}$

Example of Chi-Square Test

- Goal: Whether $X \sim \mathrm{N}(0,1)$, where $X = \left(U^{0.135} - (1-U)^{0.135}\right)/0.1975$
- Sample: Generate a sample of size $n = 1000$, divide into $b = 10$ bins.
- Chi-Square Test: $\sum_{i=1}^{10} \frac{(e_i - o_i)^2}{e_i} = 11.08$. $b - m - 1 = 10 - 0 - 1 = 9$
  p-value: $\mathrm{P}(\chi^2 > 11.08) > 0.25$. So, do not reject $H_0 : \{X_i\}_{i=1}^n \sim N(0,1)$

Bin No. i	Intervals	e_i	o_i	$\frac{(e_i - o_i)^2}{o_i}$
1	$(-\infty, -1.282]$	100	112	1.44
2	$(-1.282, -0.842]$	100	100	0
3	$(-0.842, -0.524]$	100	107	0.49
4	$(-0.524, -0.253]$	100	78	4.84
5	$(-0.253, 0]$	100	87	1.69
6	$(0, 0.253]$	100	101	0.01
7	$(0.253, 0.524]$	100	101	0.01
8	$(0.524, 0.842]$	100	98	0.04
9	$(0.842, 1.282]$	100	100	0
10	$(1.282, \infty)$	100	116	2.56
		1000	1000	11.08

Example of K-S Test

- Goal: Whether $X \sim \mathrm{N}(0, 1)$, where $X = \left(U^{0.135} - (1-U)^{0.135}\right)/0.1975$
- K-S Test:
  - Generate data $n = 1000$ from MSExcel, or Minitab, or R
  - Compute $D(n) = \max_{1 \le i \le n} |F_X(x_{(i)}) - \hat{F}_X(x_{(i)})|$, where $\hat{F}_X(x_{(i)}) = i/n$
  - Compute p-value
- Conclusion: $D(n) = 0.023$. KS Table shows: $K_{0.05}(n = 1000) = 0.04$, $K_{0.2}(n = 1000) = 0.0338$. So the p-value is greater than 0.2. That is, We do not reject $H_0 : \{X_i\}_{i=1}^{n} \sim N(0, 1)$

HW: Goodness-of-Fit Test

- Goal:
  (1) Check whether $Z \sim \mathrm{N}(0, 1)$, where $Z = \left(U^{0.135} - (1-U)^{0.135}\right)/0.1975$
  (2) Check whether $Y = Z^2 \sim \chi^2$(d.f =1), where Z is defined in (1)
- Sample: Generate data $n = 100$ from MSExcel, or Minitab, or R
- Chi-Square Test: Choose b such that each bin contains at least 5 observations.
- K-S Test:

Hint:

- Q: How do we get $\phi(z)$?
- Minitab: Calc $\longrightarrow$ Normal Dist. $\longrightarrow$Inverse Cumulative Prob. $\longrightarrow$ Inverse Input Constant (say z)
- MSExcel: Norm.inv(z, 0, 1)