Chapter 4 Vehicle Dynamics

4.1. Introduction

In order to design a controller, a good representative model of the system is needed. A vehicle mathematical model, which is appropriate for both acceleration and deceleration, is described in this section. This model will be used for design of control laws and computer simulations. Although the model considered here is relatively simple, it retains the essential dynamics of the system.

4.2. System Dynamics

The model identifies the wheel speed and vehicle speed as state variables, and it identifies the torque applied to the wheel as the input variable. The two state variables in this model are associated with one-wheel rotational dynamics and linear vehicle dynamics. The state equations are the result of the application of Newton's law to wheel and vehicle dynamics.

4.2.1. Wheel Dynamics

The dynamic equation for the angular motion of the wheel is

$$\dot{\omega}_w = [T_e - T_b - R_w F_t - R_w F_w] / J_w \qquad (4.1)$$

where J_w is the moment of inertia of the wheel, ω_w is the angular velocity of the wheel, the overdot indicates differentiation with respect to time, and the other quantities are defined in Table 4.1.

Table 4.1. Wheel Parameters

R_w	Radius of the wheel
N_v	Normal reaction force from the ground
T_e	Shaft torque from the engine
T_b	Brake torque
F_t	Tractive force
F_w	Wheel viscous friction

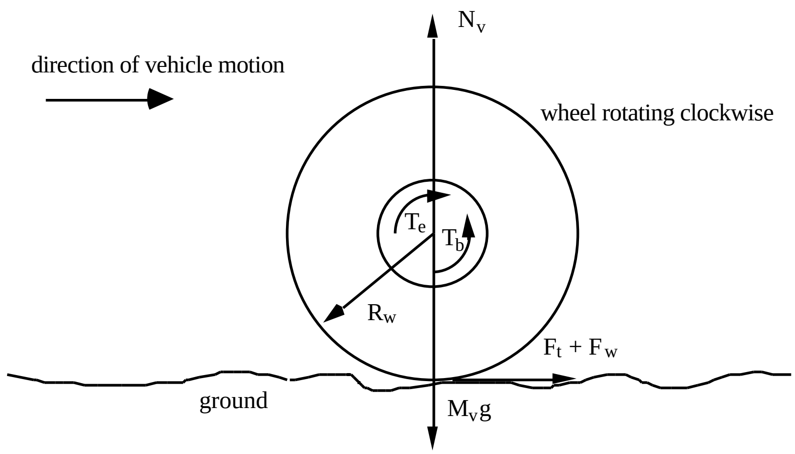

Figure 4.1. Wheel Dynamics (under the influence of engine torque, brake torque, tire tractive force, wheel friction force, normal reaction force from the ground, and gravity force)

The total torque acting on the wheel divided by the moment of inertia of the wheel equals the wheel angular acceleration (deceleration). The total torque consists of shaft torque from the engine, which is opposed by the brake torque and the torque components due to the tire tractive force and the wheel viscous friction force.

The tire tractive (braking) force is given by

$$F_t = \mu(\lambda) N_v \qquad (4.2)$$

where the normal tire force (the reaction force from the ground to the tire), Nv, depends on vehicle parameters such as the mass of the vehicle, location of the center of gravity of the vehicle, and the steering and suspension dynamics.

Applying a driving torque or a braking torque to a pneumatic tire produces tractive (braking) force at the tire-ground contact patch. The driving torque produces compression at the tire tread in front of and within the contact patch. Consequently, the tire travels a shorter distance than it would if it were free rolling. In the same way, when a braking torque is applied, it produces tension at the tire tread within the contact patch and at the front. Because of this tension, the tire travels a larger distance than it would if it were free rolling. This phenomenon is referred as the wheel slip or deformation slip (Wong, 1978). The adhesion coefficient, which is the ratio between the tractive (braking) force and the normal load, depends on the road-tire conditions and the value of the wheel slip λ (Harnel,1969). Figure 4.2. shows a typical $\mu(\lambda)$ curve. Mathematically, wheel slip is defined as

$$\lambda = (\omega_W - \omega_V) / \omega, \omega \neq 0 \quad (4.3)$$

where $\omega_V = \frac{V}{R_W}$ is the vehicle angular velocity of the wheel which is defined as being equal to the linear vehicle velocity, V, divided by the radius of the wheel. The variable ω is defined as

$$\omega = \max(\omega_W, \omega_V) \quad (4.4)$$

which is the maximum of the vehicle angular velocity and wheel angular velocity.

The adhesion coefficient $\mu(\lambda)$ is a function of wheel slip λ. For various road conditions, the $\mu(\lambda)$ curves have different peak values and slopes, as shown in Figure 4.3. In our simulation (see Chapter 5), the function $\mu(\lambda) = \frac{2\mu_p \lambda_p \lambda}{\lambda_p^2 + \lambda^2}$ is used for a nominal curve, where μ_p and λ_p are the peak values. For various road conditions, the curves have different peak values and slopes (see Figure 4.1. and Table 4.2.). The adhesion coefficient slip characteristics are also influenced by operational parameters such as speed and vertical load. The peak value for the adhesion coefficient usually has values between 0.1 (icy road) and 0.9 (dry asphalt and concrete).

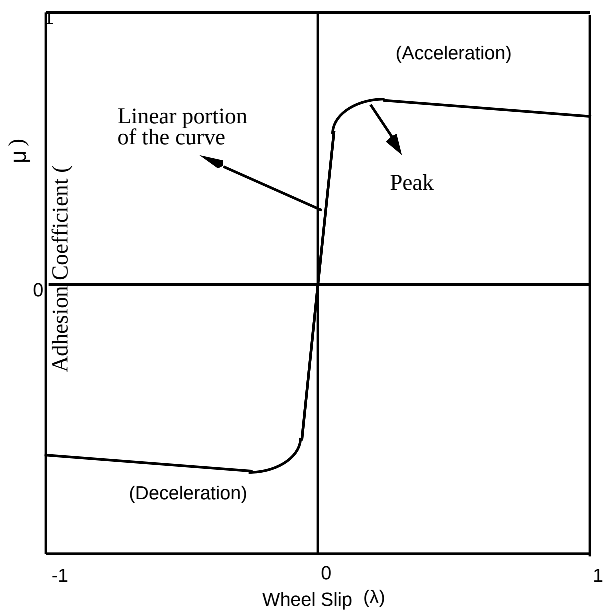

Figure 4.2. Typical μ-λ curve.

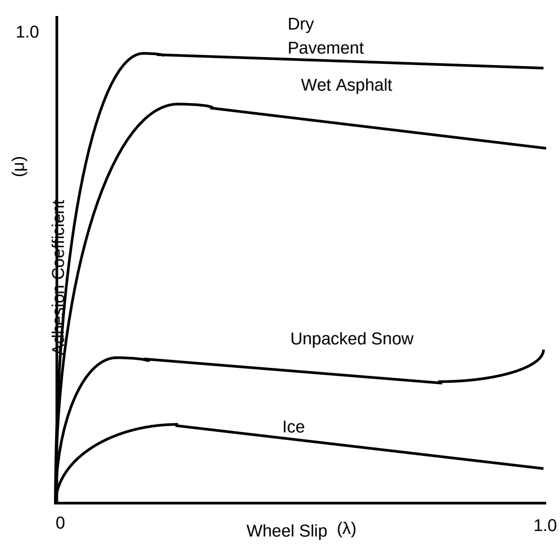

Figure 4.3. μ-λ Curves for Different Road Conditions.

Table 4.2. Average peak values for friction coefficient for different road conditions.

Surface	Average Peak
Asphalt and concrete (dry)	0.8-0.9
Asphalt (wet)	0.5-0.6
Concrete (wet)	0.8
Earth road (dry)	0.68
Earth road (wet)	0.55
Gravel	0.6
Ice	0.1
Snow (hard packed)	0.2

4.2.2. Vehicle Dynamics

The dynamic equation for the vehicle motion is

$$\dot{V} = [N_w F_t - F_v] / M_v. \tag{4.5}$$

where F_v = wind drag force (function of vehicle velocity), M_v = vehicle mass, N_w = number of driving wheels (during acceleration) or the total number of wheels (during braking), and F_t = tire tractive force, which is the average friction force of the driving wheels for acceleration and the average friction force of all wheels for deceleration. The linear acceleration of the vehicle is equal to the difference between the total tractive force available at the tire-road contact and the aerodynamic drag on the vehicle, divided by the mass of the vehicle. The total tractive force is equal to the product of the average friction force, Ft, and the number of wheels, N_w. The aerodynamic drag is a nonlinear function of the vehicle velocity and is highly dependent on weather conditions (Kachroo 1992). It is usually proportional to the square of the vehicle velocity.

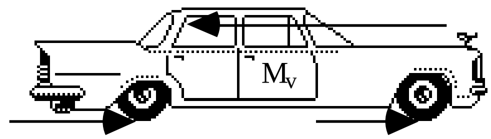

Figure 4.4. Vehicle Dynamics

4.2.3. Combined System

The dynamic equation of the whole system can be written in state variable form by defining convenient state variables. Using equations (4.1) and (4.5), and defining the state variables as

$$x_1 = \frac{V}{R_w} \tag{4.6}$$

$$x_2 = \omega_w \tag{4.7}$$

and denoting $x = \max(x_1, x_2)$, we obtain

$$\dot{x}_1 = -f_1(x_1) + b_{1N}\mu(\lambda) \tag{4.8}$$

$$\dot{x}_2 = -f_2(x_2) - b_{2N}\mu(\lambda) + b_3 T \tag{4.9}$$

where

$$T = T_e - T_b$$

$$\lambda = (x_2 - x_1)/x$$

$$f_1(x_1) = [F_v(R_w x_1)]/(M_v R_w)$$

$$b_{1N} = N_v N_w /(M_v R_w)$$

$$f_2(x_2) = F_w(x_2)/J_w$$

$$b_{2N} = R_w N_v / J_w$$

$$b_3 = 1/J_w \tag{4.10}$$

The combined dynamic system can be represented as shown in Figure 4.5. The control input is the applied torque at the wheels, which is equal to the difference between the shaft torque from the engine and the braking torque.

During acceleration, engine torque is the primary input while during deceleration, the braking torque is the primary input.

4.2.4. System Dynamics in Terms of Wheel Slip

Wheel slip is chosen as the controlled variable for braking control algorithms because of its strong influence on the braking force between the tire and the road. By controlling the wheel slip, we control the braking force to obtain the desired output from the system. In order to control the wheel slip, we can have system dynamic equations in terms of wheel slip. During deceleration, condition $x_2 \le x_1$, ($x_1 \neq 0$) is satisfied, and therefore wheel slip is defined as:

$$\lambda = (x_2 - x_1) / x_1 \tag{4.11}$$

Differentiating this equation, we obtain

$$\dot{\lambda} = [\dot{x}_2 - (1+\lambda)\dot{x}_1] / x_1 \tag{4.12}$$

Substituting equations (4.8) (4.9) and (4.11) into equation (4.12), we obtain

$$\dot{\lambda} = [(1+\lambda) f_1(x_1) - f_2(x_2)] - [b_{2N} + (1+\lambda) b_{1N}]\mu + b_3 T] / x_1 \tag{4.13}$$

This gives the wheel slip dynamic equation for deceleration. This equation is nonlinear and involves uncertainties in its parameters. The nonlinear characteristics of the equation are due to the following:

- the relationship of wheel slip with velocity is nonlinear,
- the μ-λ relationship is nonlinear,
- there are multiplicative terms in the equation,
- functions $f_1(x_1)$ and $f_2(x_2)$ are nonlinear.

The uncertainties of the parameters are due to the following:

- N_v (normal tire force) changes based on steering and suspension dynamics,
- the μ-λ curve changes based on road surface,

- moment of inertia of the wheel is changing all the time,
- wind drag force changes based on uncertain wind speed and direction.

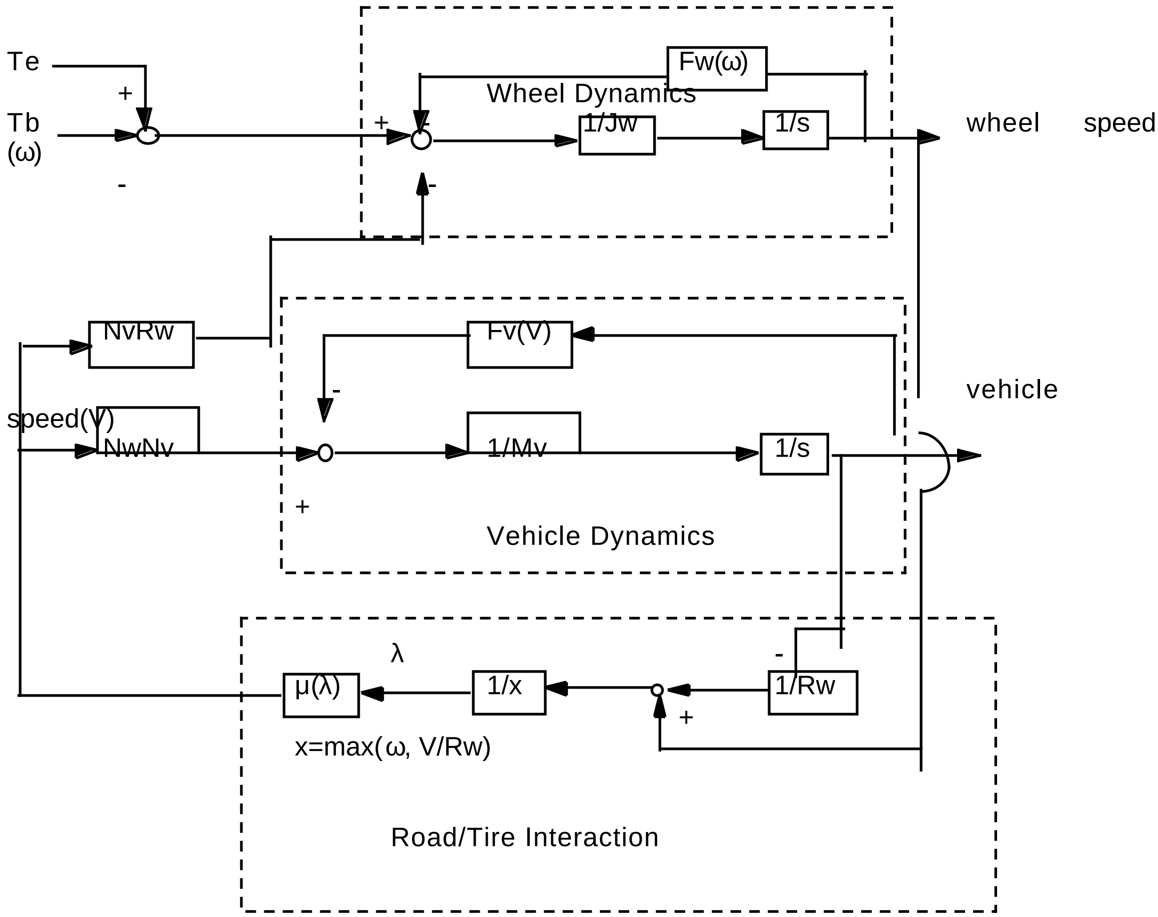

Figure 4.5 Vehicle/road/tire one-wheel model

4.2.5. Stability of the Dynamic System

Before we design a control mechanism for the combined system, we want to assure the stability of the system. Due to the nonlinearity of the system, local stability could be evaluated.

The local stability of a nonlinear system can be studied by linearizing the system around its equilibrium point. For general differential equations in the form of

$$\dot{x}(t) = f(x(t), u(t)) \quad (4.14)$$

where x(t) is the state vector, u(t) is the input vector, and f: $R^n \times R^m \rightarrow R^n$. The pair ($x_0$, u_0) is called an equilibrium if f(x_0, u_0) = 0 . Starting from the initial condition x(0) = x_0 with a constant input u(t)= u_0, the solution remains at x(t)= x_0.

Assuming that f is continuously differentiable at (x_0, u_0), a multivariable Taylor

series expansion yields

$$\dot{x}(t) = f(x0,\ u0) + D_1 f(x_0, u_0)(x(t) - x_0)$$

$$+ D_2 f(x0,\ u0)\ (u(t) - u_0) + r(x(t), u(t)) \quad (4.15)$$

where the remainder, r(x,u), satisfies

$$\lim_{(x,u) \to (x_0, u_0)} \frac{r(x,u)}{\sqrt{|x - x_0|^2 + |u - u_0|^2}} = 0 \quad (4.16)$$

By defining the deviation-from-equilibrium terms

$$\tilde{x}(t) = x(t) - x_0,\ \tilde{u}(t) = u(t) - u_0 \quad (4.17)$$

and assuming that the equilibrium is fixed, along with the equilibrium condition f(x_0, u_0) = 0, the following equation represents the linearization of the nonlinear dynamics about the equilibrium point (x_0, u_0):

$$\dot{\tilde{x}} \quad D_1 f(x_0, u_0)\tilde{x}(t) + D_2 f(x_0, u_0)\tilde{u}(t) \quad (4.18)$$

where $D_1 f$ and $D_2 f$ denote the Jacobian matrices with respect to the first variable (x) and the second variable (u).

The linearization of a nonlinear system can be used to analyze stability. Let x_0 be an equilibrium for the unforced state equations (u(t)=0)

$$\dot{x}(t) = f(x(t), 0) \quad (4.19)$$

That means f(x_0) =0. The equilibrium x_0 is stable if for each $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$|x(0) - x_0| < \delta(\varepsilon) \qquad |x(t) - x_0| < \varepsilon, \quad t \geq 0 \quad (4.20)$$

and it is asympototically stable if it is stable and for some δ_1

$$|x(0) - x_0| < \delta_1 \qquad |x(t) - x_0| \to 0, \text{ as } t \to \infty$$

(4.21)

Let $f\colon R^n \to R^n$ be continuously differentiable. The equalibrium x_0 is asymptotically stable if all of the eigenvalues of $Df(x_0)$ have strictly negative real parts. It is unstable if $Df(x_0)$ has an eigenvalue with a positive real part. As a result, linearization can provide sufficient conditions for stability of the nonlinear system in a sufficiently small neighborhood of an equilibrium and only if the system is time-invariant. Only in the case that $Df(x_0)$ has purely imaginary eigenvalues, then nonlinear methods have to be used to evaluate the stability of the nonlinear system.

The vehicle nonlinear system equations are linearized around the equilibrium point in order to study the system stability. The equilibrium point (x_{10}, x_{20}) of the vehicle system described by Equations (4.8) and (4.9) can be obtained by equating the right hand sides of the two equations to zero. Then the Jacobian matrix can be evaluated to assess the stability of the system.

For the deceleration case, the Jacobian matrix at the equilibrium is:

$$A = \begin{bmatrix} -\frac{df_1}{dx_1}(x_{10}) - b_{1N}\frac{\partial\mu}{\partial\lambda}(x_{10},x_{20})\frac{x_{20}}{x_{10}^2}, & b_{1N}\frac{\partial\mu}{\partial\lambda}(x_{10},x_{20})\frac{1}{x_{10}} \\ b_{2N}\frac{\partial\mu}{\partial\lambda}(x_{10},x_{20})\frac{x_{20}}{x_{10}^2}, & -\frac{df_2}{dx_2}(x_{20}) - b_{2N}\frac{\partial\mu}{\partial\lambda}(x_{10},x_{20})\frac{1}{x_{10}} \end{bmatrix} \quad (4.22)$$

The eigenvalues of A are obtained by solving for λ_e in the equation

$$\det(\lambda_e I - A) = 0$$

The real part of the eigenvalues of A are calculated to be

$$\frac{-\left[\frac{df_1}{dx_1}(x_{10}) + \frac{df_2}{dx_2}(x_{20}) + \frac{\partial\mu}{\partial\lambda}(x_{10},x_{20})[b_{1N}\frac{x_{20}}{x_{10}^2} + b_{2N}\frac{1}{x_{10}}]\right]}{2}$$

Here also df_1/dx_1, df_2/dx_2, x_1, x_2, b_{1N} and b_{2N} are all positive (see equations in (4.10)), so when $\partial\mu/\partial\lambda$ is positive, the eigenvalues of A have negative real parts. When $\partial\mu/\partial\lambda$ is negative, the eigenvalues of A have positive real parts for

$$\left[b_{1N} \frac{x_{20}}{x_{10}^2} + b_{2N}/x_{10} \right] > \frac{\frac{df_1}{dx_1}(x_{10}) + \frac{df_2}{dx_2}(x_{20})}{\left|\frac{\partial\mu}{\partial\lambda}\right|} \qquad (4.23)$$

Therefore, only under condition (4.23) the system is unstable.

4.3. Summary

In this section, the dynamics of the combined system of wheel and vehicle are described. The condition to assure the system stability is also described. The vehicle model described by (4.8), (4.9) and (4.10) will be used to represent the plant in simulation and control system design for the rest of the thesis.