Magnetically Coupled Circuits

- Whenever a current flows through a conductor, a magnetic field is generated (magnetic flux)
- When time varying magnetic field generated by one loop penetrates a second loop, a voltage induced between the ends of the second wire
- New term is defined “mutual inductance” to differentiate from “inductance”

MUTUAL INDUCTANCE

Physical basis

- Production of magnetic flux by a current
- The production of voltage by time varying magnetic field
- Current flowing in one coil established flux about that coil and about second coil nearby
- The time varying flux surrounding second coil produces a voltage across the terminals of the second coil
- This voltage is proportional to time rate of change of current in first coil
- Define coefficient of mutual inductance or simply mutual inductance
\[ v_2(t) = M_{21} \frac{di_1(t)}{dt} \]

**FIGURE 13.1** (a) A current \( i_1 \) through \( L_1 \) produces an open-circuit voltage \( v_2 \) across \( L_2 \). (b) A current \( i_2 \) through \( L_2 \) produces an open-circuit voltage \( v_1 \) across \( L_1 \).

\[ v_1(t) = M_{12} \frac{di_2(t)}{dt} \]

- \( M_{12} = M_{21} = M \)
Dot Convention

- A current entering the dotted terminal of one coil produces an open circuit voltage with positive voltage reference at the dotted terminal of the second coil.
- A current entering the undotted terminal of one coil provides a voltage that is positively sensed at the undotted terminal of the second coil.
Example

For the circuit shown in Fig. 13.3, (a) determine $v_1$ if $i_2 = 5 \sin 45t$ A and $i_1 = 0$; (b) determine $v_2$ if $i_1 = -8e^{-t}$ A and $i_2 = 0$.

(a) Since the current $i_2$ is entering the undotted terminal of the right coil, the positive reference for the voltage induced across the left coil is the undotted terminal. Thus, we have an open-circuit voltage

$$v_1 = -(2)(45)(5 \cos 45t) = -450 \cos 45t \text{ V}$$

appearing across the terminals of the left coil as a result of the time-varying magnetic flux generated by $i_2$ flowing into the right coil.

(Continued on next page)

Since no current flows through the coil on the left, there is no contribution to $v_1$ from self-induction.

(b) We now have a current entering a dotted terminal, but $v_2$ has its positive reference at the undotted terminal. Thus,

$$v_2 = -(2)(-1)(-8e^{-t}) = -16e^{-t} \text{ V}$$
Combined Mutual and Self-Induction Voltage

\[ v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]
\[ v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \]

\[ v_1 = -L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]
\[ v_2 = -L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \]

For sinusoidal source

\[ V_1 = -j\omega L_1 I_1 + j\omega M I_2 \]
\[ V_2 = -j\omega L_2 I_2 + j\omega M I_1 \]
Physical Basis of the Dot Convention

- The current $i_1$ produces a flux which directed downward.
- Since $i_1$ is increasing with time, the flux, which is proportional to $i_1$ is also increasing with time.
- If $i_2$ is a positive and increasing, $i_2$ produces a magnetic flux which is directed downward and increasing.
- The voltage across the terminals of any coil results from the time rate of change of the flux within the coil.
- The voltage is greater with $i_2$ flowing than it would be if $i_2$ were zero.
Example

For the circuit shown in Fig. 13.7a, find the ratio of the output voltage across the 400 Ω resistor to the source voltage, expressed using phasor notation.

![Figure 13.7](a) A circuit containing mutual inductance in which the voltage ratio \( V_2/V_1 \) is desired. (b) Self and mutual inductances are replaced by the corresponding impedances.

1. **Identify the goal of the problem.**
   We need a numerical value for \( V_2 \). We will then divide by 10/0° V.

2. **Collect the known information.**
   We begin by replacing the 1 H and 100 H by their corresponding impedances, \( j/10 \) Ω and \( jk \) Ω, respectively (Fig. 13.7b). We also replace the 9 H mutual inductance by \( j\omega M = j90 \) Ω.

3. **Devise a plan.**
   Mesh analysis is likely to be a good approach, as we have a circuit with two clearly defined meshes. Once we find \( I_3 \), \( V_2 \) is simply 400 \( I_3 \).

4. **Construct an appropriate set of equations.**
   In the left mesh, the sign of the mutual term is determined by applying the dot convention. Since \( I_3 \) enters the undotted terminal of \( L_2 \), the mutual voltage across \( L_1 \) must have the positive reference at the undotted terminal. Thus,
   \[
   (1 + j10)I_1 - j90I_2 = 10/0°.
   \]
   Since \( I_1 \) enters the dot-marked terminal, the mutual term in the right mesh has its (+) reference at the dotted terminal of the 100 H inductor. Therefore, we may write
   \[
   (400 + j1000)I_2 - j90I_1 = 0
   \]

5. **Determine if additional information is required.**
   We have two equations in two unknowns, \( I_1 \) and \( I_2 \). Once we solve for the two currents, the output voltage \( V_2 \) may be obtained by multiplying \( I_2 \) by 400 Ω.

6. ** Attempt a solution.**
   Upon solving these two equations with a scientific calculator, we find that
   \[
   I_2 = 0.172/\times 16.70° \text{ A}
   \]
   Thus,
   \[
   \frac{V_2}{V_1} = \frac{400(0.172/\times 16.70°)}{10/0°} = 6.880/\times 16.70°
   \]
Example

Write a complete set of phasor equations for the circuit of Fig. 13.10a.

The circuit contains three meshes, and the three mesh currents have already been assigned. Once again, our first step is to replace both the mutual inductance and the two self-inductances with their corresponding impedances as shown in Fig. 13.10b. Applying Kirchhoff’s voltage law to the first mesh, a positive sign for the mutual term is assured by selecting \((I_1 - I_2)\) as the current through the second coil. Thus,

\[
5I_1 + 7j\omega(I_1 - I_2) + 2j\omega(I_3 - I_3) = V_1
\]

or

\[
(5 + 7j\omega)I_1 - 9j\omega I_2 + 2j\omega I_3 = V_1 \tag{3}
\]

The second mesh requires two self-inductance terms and two mutual-inductance terms; the equation cannot be written carelessly. We obtain

\[
7j\omega(I_2 - I_1) + 2j\omega(I_2 - I_3) + \frac{1}{j\omega}I_1 + 6j\omega(I_2 - I_3)
+ 2j\omega(I_2 - I_1) = 0
\]

or

\[
-9j\omega I_1 + \left(17j\omega + \frac{1}{j\omega}\right)I_2 - 8j\omega I_3 = 0 \tag{4}
\]

Finally, for the third mesh,

\[
6j\omega(I_3 - I_1) + 2j\omega(I_3 - I_2) + 3I_3 = 0
\]

or

\[
2j\omega I_1 - 8j\omega I_2 + (3 + 6j\omega)I_3 = 0 \tag{5}
\]

Equations \([3]\) to \([5]\) may be solved by any of the conventional methods.
Primary
Secondary

Reflected Impedance

\[ V_s = (R_1 + j \omega L_1)I_1 - j \omega MI_2 \]
\[ 0 = -j \omega M I_1 + (R_2 + j \omega L_2 + Z_L)I_2 \]

Define

\[ Z_{11} = R_1 + j \omega L_1 \quad \text{and} \quad Z_{22} = R_2 + j \omega L_2 + Z_L \]

\[ V_s = Z_{11}I_1 - j \omega MI_2 \]
\[ 0 = -j \omega M I_1 + Z_{22}I_2 \]

\[ Z_{in} = \frac{V_s}{I_1} = Z_{11} - \frac{(j \omega)^2 M^2}{Z_{22}} \]

\[ Z_{in} = Z_{11} + \frac{\omega^2 M^2}{R_{22} + j X_{22}} \]

\[ Z_{in} = Z_{11} + \frac{\omega^2 M^2 R_{22}}{R_{22}^2 + X_{22}^2} + \frac{-j \omega^2 M^2 X_{22}}{R_{22}^2 + X_{22}^2} \]

- Presence of secondary increases losses in primary
T Equivalent Network

\[ v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} \]

\[ v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} \]
Find the T equivalent of the linear transformer shown in Fig. 13.19a.

We identify $L_1 = 30 \, \text{mH}$, $L_2 = 60 \, \text{mH}$, and $M = 40 \, \text{mH}$, and note that the dots are both at the upper terminals, as they are in the basic circuit of Fig. 13.17.

Hence, $L_1 - M = -10 \, \text{mH}$ is in the upper left arm, $L_2 - M = 20 \, \text{mH}$ is at the upper right, and the center stem contains $M = 40 \, \text{mH}$.

The complete equivalent T is shown in Fig. 13.19b.

To demonstrate the equivalence, let us leave terminals C and D open-circuited and apply $V_{AB} = 10 \cos 100\omega \, \text{V}$ to the input in Fig. 13.19a. Thus,

$$i_1 = \frac{1}{30 \times 10^{-3}} \int 10 \cos(100\omega) \, dt = 3.33 \sin 100\omega \, \text{A}$$

and

$$v_{CD} = M \frac{di_1}{dt} = 40 \times 10^{-3} \times 3.33 \times 100 \cos 100\omega$$
$$= 13.33 \cos 100\omega \, \text{V}$$

Applying the same voltage in the T equivalent, we find that

$$i_1 = \frac{1}{(-10 + 40) \times 10^{-3}} \int 10 \cos(100\omega) \, dt = 3.33 \sin 100\omega \, \text{A}$$

once again. Also, the voltage at C and D is equal to the voltage across the 40 mH inductor. Thus,

$$v_{CD} = 40 \times 10^{-3} \times 3.33 \times 100 \cos 100\omega = 13.33 \cos 100\omega \, \text{V}$$

and the two networks yield equal results.
**IDEAL TRANSFORMER**

Ideal transformer is a useful approximation of a very tightly coupled transformer in which coupling coefficient is unity and both secondary and primary inductive reactance are extremely large.

<table>
<thead>
<tr>
<th>Turns ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \frac{L_2}{L_1} = \frac{N_2^2}{N_1^2} = a^2 ]</td>
</tr>
<tr>
<td>[ a = \frac{N_2}{N_1} ]</td>
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</tbody>
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**Sinusoidal Steady State**

\[ V_1 = j\omega L_1 I_1 - j\omega M I_2 \]

\[ 0 = -j\omega M I_1 + (Z_L + j\omega L_2)I_2 \]

\[ V_1 = I_1 j\omega L_1 + I_1 \frac{\omega^2 M^2}{Z_L + j\omega L_2} \]

\[ Z_{in} = \frac{V_1}{I_1} = j\omega L_1 + \frac{\omega^2 M^2}{Z_L + j\omega L_2} \]

\[ Z_{in} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{Z_L + j\omega L_2} \]
\[ L_2 = a^2 L_1 \]

\[ Z_{in} = j \omega L_1 + \frac{\omega^2 a^2 L_1^2}{Z_L + j \omega a^2 L_1} \]

\[ Z_{in} = \frac{j \omega L_1 Z_L - \omega^2 a^2 L_1^2 + \omega^2 a^2 L_1^2}{Z_L + j \omega a^2 L_1} \]

\[ Z_{in} = \frac{j \omega L_1 Z_L}{Z_L + j \omega a^2 L_1} = \frac{Z_L}{Z_L / j \omega L_1 + a^2} \]

As \( L_1 \to \infty \)

\[ Z_{in} = \frac{Z_L}{a^2} \]
Use of Transformers for Impedance Matching

Example:

Match amplifier internal impedance of 4000\(\Omega\) and a speaker impedance of 8\(\Omega\).

\[
\begin{align*}
Z_g &= 4000 = \frac{Z_L}{a^2} = \frac{8}{a^2} \\
\Rightarrow a &= \frac{1}{22.4} \\
\frac{N_1}{N_2} &= 22.4
\end{align*}
\]

Primary and secondary current relationship

\[
\frac{I_2}{I_1} = \frac{j\omega M}{Z_L + j\omega L_2}
\]

If \(L_2\) is very large and since \((M=(L_1L_2)^{1/2})\)

\[
\frac{I_2}{I_1} = \frac{j\omega M}{j\omega L_2} = \sqrt{\frac{L_1}{L_2}}
\]

\[
\frac{I_2}{I_1} = \frac{1}{a}
\]

\[
N_1I_1 = N_2I_2
\]
Use of Transformers for Voltage Level Adjustment

\[ V_2 = I_2 Z_L \]

\[ V_1 = I_1 Z_{in} = I_1 \frac{Z_L}{a^2} \]

\[ V_1 = I_1 Z_{in} = I_1 \frac{Z_L}{a^2} \]

\[ \frac{V_2}{V_1} = a^2 \frac{I_2}{I_1} \]

\[ \frac{V_2}{V_1} = a = \frac{N_2}{N_1} \]

\[ V_2 I_2 = V_1 I_1 \]

\[ Z_L = \frac{|Z_L|}{\theta} \]