1. Consider a communication system where three equiprobable messages \( m_1, m_2, \) and \( m_3 \) are transmitted. Let \( m_1, m_2, \) and \( m_3 \) be encoded by signals \( s_1(t), s_2(t), \) and \( s_3(t), \) respectively, given by

\[
\begin{align*}
  s_1(t) &= 2 \sqrt{2} \cos 2\pi t \\
  s_2(t) &= 2 \sqrt{2} \sin 2\pi t \\
  s_3(t) &= -2 \sqrt{2} \sin 2\pi t
\end{align*}
\]

where the signal duration is \( 0 \leq t \leq 1 \) and each signal is zero outside this interval. Assume that the signals are transmitted over an additive white Gaussian noise channel.

(a) Find a set of orthonormal basis functions to represent the set of signals, and then draw the corresponding signal constellation. (5%)

(b) Determine the optimum decision regions. (5%)

(c) Determine an equivalent minimum-energy signal set that would yield the same probability of error as the signal set described above. Draw the corresponding signal constellation and optimum decision regions. (5%)

2. Consider a coherent binary frequency-shift keying (FSK) system where symbols 1 and 0 occur with equal probability. Let symbols 1 and 0 be encoded by signals \( s_1(t) \) and \( s_2(t), \) respectively, given by

\[
s_i(t) = \begin{cases} 
\sqrt{2E_s/T_0} \cos(2\pi f_c t), & 0 \leq t \leq T_0 \\
0, & \text{elsewhere}
\end{cases}
\]

where \( i = 1, 2, \) \( E_s \) is the transmitted signal energy per bit, \( T_0 \) is the symbol duration, and \( f_c = (n+1)/T_0 \) for some fixed integer \( n. \) Assume that a white Gaussian noise process of zero mean and power spectral density \( N_0/2 \) is added during the transmission of an FSK signal.

(a) Determine the optimum receiver. (6%)

(b) Derive the error probability of the optimum receiver in terms of the complementary error function defined by

\[
\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2) \, dz. 
\]

(You must give derivations, or you would get no points in this sub-problem!)
3. Consider the following random-phase sinusoidal process

\[ x(t) = A \cos(\omega_0 t + \theta), \quad -\infty < t < \infty \]

where \( \omega_0 \) is a constant, \( \theta \) is a random variable uniformly distributed over \([0, 2\pi]\) and \( A \) is a binary random variable with probabilities \( P_1[A = 1] = p \) and \( P_1[A = 2] = 1 - p \).

(a) Find the mean and correlation function of \( x(t) \). (10%) 

(b) Hilbert transformer is a linear time-invariant system with frequency response

\[ H(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases} \]

Assume that \( x(t) \) is input to the Hilbert transformer and \( y(t) \) is the associated output. Find the power spectral density of \( y(t) \). (5%) 

4. Assume that

\[ x(t) = ax(t) + n(t), \quad -\infty < t < \infty \]

where \( n(t) \) is white Gaussian noise with zero mean and power spectral density \( S_n(f) = 1 \), and the waveform of the signal \( s(t) \) is given by

\[ s(t) = \begin{cases} 1-t, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \]

(a) Find the matched filter impulse response and peak output signal squared to output noise variance. (8%) 

(b) Assume that \( a = 1 \) or \( a = -1 \) with equal prior probability and that \( y(t_e) \) is the matched filter output with the peak signal squared to output noise variance. Find the probability of error \( P_e \) of the detector that decides \( a = 1 \) if \( y(t_e) > 0 \) and \( a = -1 \) if \( y(t_e) < 0 \). (You can express \( P_e \) in terms of the Q-function \( Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty \exp(-z^2/2)dz \) or the complementary error function erfc(u).) (7%) 

5. For a linear modulation with the in-phase component \( s_i(t) = \frac{1}{2} m(t) \) and the quadrature component \( s_q(t) = \frac{1}{2} \hat{m}(t) \) (\( \hat{m}(t) \) = Hilbert transform of \( m(t) \)), please show that \( s(t) = s_i(t) \cos(2\pi ft) - s_q(t) \sin(2\pi ft) \) is a single-sideband (SSB) with upper sideband transmitted signal. (10%)
6. Consider that a double sideband-suppressed carrier (DSB-SC) modulated signal 
   \( s(t) = A \cos(2\pi f_c t) m(t) \) is transmitted over an additive white Gaussian noise
   channel with power spectral density \( N_0/2 \), where \( m(t) \) is the message signal with
   average message power \( P \) and message bandwidth \( W \).

   (a) Find the average power of DSB-SC modulated signal \( s(t) \) and the average noise
       power in the message bandwidth. (5%)

   (b) Find the output signal of a coherent detector. (5%)

   (c) Find the output signal-to-noise ratio. (5%)

7. Assume that a linear time-invariant filter of impulse response \( h(t) \) (frequency
   response \( H(\Omega) \)) is driven by a stationary random process \( X(t) \) (with power spectral
   density \( S_X(\Omega) \)) and \( Y(t) \) is the associated output.

   (a) Find the power spectral density \( S_Y(\Omega) \) of \( Y(t) \). (10%)

   (b) Show that the power spectral density \( S_Y(\Omega) \) is always nonnegative. (5%)