1. Find the solution $(x_1(t), x_2(t), x_3(t))$ of the system [20 points]

$$\begin{cases} x_1' &= 4x_1 + 3x_2 + x_3 \\ x_2' &= -4x_1 - 4x_2 - 2x_3 \\ x_3' &= 8x_1 + 12x_2 + 6x_3 \end{cases}$$

   with the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 1, -4)$.

2. Let $(x_1(t), x_2(t), x_3(t))$ satisfy the system [20 points]

$$\begin{cases} x_1' &= -2x_1 + x_2x_3 \\ x_2' &= x_1 - x_1x_3 \\ x_3' &= x_1x_2. \end{cases}$$

   Construct one Liapunov function to show that the origin is stable. Is the origin asymptotically stable?

3. Suppose that $A$ is a nilpotent $k \times k$ matrix for some $k \geq 100$. [20 points]
   (a) What are the eigenvalues if $A$ is a $200 \times 200$ matrix?[5 points]
   (b) Suppose that $A$ satisfies the equation

$$A^{100} = \sum_{i=1}^{100} c_i A^{i-1},$$

   for some $c_i \in \mathbb{R}$. Show that $A^{100}$ is a zero matrix.[15 points]

4. Find the general solution of [20 points]

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^2.$$

5. Suppose that $y(x)$ satisfies [20 points]

$$\frac{dy}{dx} \leq x^{-1}y + x \text{ for } x \geq 2,$$

   and $y(2) \leq 4$. Prove that $y \leq x^2$ for $x \geq 2$.

試題隨卷繳回